J Mol Model (2013) 19:2007–2014
DOI 10.1007/s00894-012-1600-3

ORIGINAL PAPER

Three–dimensional effective mass Schrödinger equation: harmonic and Morse-type potential solutions

G. Ovando · J. Morales · J. L. López-Bonilla

Received: 30 April 2012 / Accepted: 19 September 2012 / Published online: 7 October 2012

Abstract In this work, a scheme to generate exact wave functions and eigenvalues for the spherically symmetric three-dimensional position-dependent effective mass Schrödinger equation is presented. The methodology is implemented by means of separation of variables and point canonical transformations that allow to recognize a radial dependent equation with important differences as compared with the one-dimensional position dependent mass problem, which has been widely studied. This situation deserves to consider the boundary conditions of the emergent problem. To obtain specific exact solutions, the methodology requires known solutions of ordinary one-dimensional Schrödinger equations. We have preferred those applications that use the harmonic oscillator and the Morse oscillator solutions.

Keyword Harmonic potential · Morse potential · Position-dependent effective mass · Three dimensional Schrödinger equation

Introduction

The position-dependent (effective) mass Schrödinger equation (PDMSE) has a theoretical foundation widely explained in the related literature and it is concerned with aspects of solid state theory [1–6], Bhomian quantum motion [7], DFT theory [8–12], Bohr Hamiltonian in nuclear theory [13, 14] and other condensed matter systems [1, 15–19]. Study of the PDMSE has attracted a continual interest focused in determining the correct order in mass and momentum operators [20–23] and also in obtaining exact solutions [24–45]. Several techniques allow to identify the appropriate couple of mass and potential functions that lead to a known solvable problem of ordinary quantum mechanics Schrödinger equation (QMSE), also named the reference problem [45]. The point canonical transformations (PCT) [24–31], Lie algebras [32–35], supersymmetry algebra, [36–39], shape invariance technique [36, 42–45], and so on, are procedures that nowadays are being improved to visualize the mentioned pair of functions among the many possibilities that exist. Difference between the potentials of the QMSE and the corresponding ones of the PDMSE is a mass depending term which is expressed in different forms by authors. In refs. [46–50] this term was reduced after a point canonical transformation scheme to the form $V_{PCT} = W^2 + W'$, for a suitable function W. The advantages of this expression were shown for the one-dimensional PDMSE with null potential [50].

On the other hand, although the one-dimensional PDMSE has attracted wide attention, this is not the case for the three-dimensional PDMSE. The three dimensional PDMSE with a central potential and determined values of the total angular momentum and its azimuthal projection has been studied for example in [51–54], where we can see its reduction to an angular equation and a radial equation that does not correspond to the one-dimensional PDMSE. This fact is also pointed out in this work and means that resolution of the three-dimensional problem through its corresponding radial equation is a new kind of Sturm-Liouville problem to be studied in its particular form. The situation is appraised in the kind of solutions reported in references previously mentioned, which are divorced from those of the one-dimensional PDMSE. However, in the scheme we propose in this work, the

G. Ovando (✉) · J. Morales
CBI—Area de Física Atómica Molecular Aplicada, Universidad Autónoma Metropolitana—Azcapotzalco,
Av. San Pablo 180, Col. Reynosa-Tamps,
02200 Mexico City, Mexico, D. F.
e-mail: gaoz@correo.azc.uam.mx

J. L. López-Bonilla
Instituto Politécnico Nacional, ESIME-Zacatenco,
Anexo Edif. 3, Col. Lindavista,
07738 Mexico City, Mexico, D. F.

mentioned radial equation is again reduced to the same form as it had in the one-dimensional case [46–50], the difference is now transferred to the relation between the mass function and the W function, to be used in the corresponding term V_{PCT}. The modifications affect not only the expression of the mass function, but also the new independent variable involved in the point canonical transformation. That feature implies a great difference between the possible solvable potentials for the one-dimensional and the three-dimensional cases. We can mention that the one-dimensional PDMSE has been solved for the Morse potential [40, 43, 44] but resolution of this particular potential is a new challenge for the three-dimensional PDMSE as it will be discussed in this work. The article is organized as follows: in The three-dimensional PDMSE with spherical symmetry we deduce the radial PDMSE by applying the separation of variables scheme. Next, we use a point canonical transformation to reduce the radial problem to a one-dimensional QMSE with a term of the form V_{PCT}. Some considerations about the boundary conditions at the origin and at infinity are included. In Solvable three-dimensional PDMSE with spherical symmetry a series of solvable pairs of mass and potential functions are studied and we mark the cases corresponding to the pairs found by previous authors. We are particularly interested in getting solutions that use as reference potentials some of the most important QMSE potentials, like the harmonic oscillator potential and the Morse potential. Our conclusions are expressed in the final section.

The three-dimensional PDMSE with spherical symmetry

Let us consider the PDMSE expressed in three dimensions through the Nabla operator

$$-\frac{\hbar^2}{2m_0}\nabla\cdot\left(\frac{1}{M(\mathbf{r})}\nabla\Psi\right)+V(\mathbf{r})\,\Psi=E\,\Psi, \tag{1}$$

where m_0 is the mass of the particle and $M(\mathbf{r})$ is a dimensionless position dependent factor named the mass function hereafter, that is, $m(\mathbf{r})=m_0M(\mathbf{r})$ is the position dependent mass of the system. Suppose that both, the mass function and the potential are only r dependent functions, then, the previous equation written in terms of the total angular momentum operator $\mathbf{L}$, is

$$\frac{\hbar^2}{2m}\left(-\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{M(r)}\frac{\partial\Psi}{\partial r}\right)+\frac{\mathbf{L}^2}{r^2M(r)}\Psi\right)+V(r)\Psi=E\,\Psi. \tag{2}$$

Spherical symmetry allows to look for eigenstates of the total angular momentum and its z - projection in the form $\Psi(r,\theta,\varphi)=R(r)Y_{lm}(\theta,\varphi)$, with $Y_{lm}(\theta,\varphi)$ being the spherical harmonics and $R(r)$ the radial function that satisfies the radial PDMSE

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{M(r)}\frac{dR(r)}{dr}\right)-\frac{l(l+1)}{r^2M(r)}R(r)+\kappa[E-V(r)]R(r)=0, \tag{3}$$

where $\frac{1}{\kappa}=\frac{\hbar^2}{2m_0}$. Without loss of generality, hereafter we set $\kappa=1$. The previous equation can also be written in the form

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{M(r)}\frac{dR(r)}{dr}\right)+[E-V_l(r)]R(r)=0, \tag{4}$$

where the potential with centrifugal term is

$$V_l(r)=V(r)+\frac{l(l+1)}{r^2M(r)}. \tag{5}$$

For the sake of simplicity, we will use only the name of the function for those which depend on the variable r and are expressible also in the variable u, for example $M=M(r)=M(r(u))$, $W=W(r)=W(r(u))$ and so on. In order to solve the previous radial PDMSE let us take the variable change

$$du=dr\sqrt{M}, \tag{6}$$

for a definitely positive function of mass M. After change of variable, Eq. 4 is

$$\frac{\sqrt{M}}{r^2}\frac{d}{du}\left(\frac{r^2}{\sqrt{M}}\frac{dR}{du}\right)+[E-V_l]R=0 \tag{7}$$

or alternatively

$$\frac{d^2R}{du^2}+\frac{d}{du}\left(ln\frac{r^2}{\sqrt{M}}\right)\frac{dR}{du}+[E-V_l]R=0. \tag{8}$$

To complete a point canonical transformation [51] we define new radial unknown function φ by

$$\varphi=R\,e^{\int W\,du}, \tag{9}$$

which allows the identity

$$\left(R''+2W\,R'\right)e^{\int W\,du}=\frac{d^2\varphi}{du^2}-W'\varphi-W^2\varphi \tag{10}$$

and transforms Eq. 8, into the form

$$\frac{d^2\varphi}{du^2}-W'\varphi-W^2\varphi+[E-V_l]\varphi=0. \tag{11}$$

This last equation is an ordinary one-dimensional Schrödinger problem

$$\frac{d^2\varphi}{du^2}+[E-U_l]\varphi=0 \tag{12}$$

with potential

$$U_l=V+\frac{l(l+1)}{r^2M}+\left(W^2+W'\right), \tag{13}$$

where function W is given by

$$W = \frac{d}{du}\ln\frac{r}{M^{1/4}} = \frac{1}{r}\frac{dr}{du} - \frac{1}{4M}\frac{dM}{du}. \tag{14}$$

Upon substitution of this W function into Eq. 9 it merges the solution of the radial PDMSE in the form

$$R(r) = \varphi\frac{M^{1/4}}{r}, \tag{15}$$

which seems to be the usual form of the standard radial function in quantum mechanics. We can mention that Eq. 14 is one of our main results as it is the three-dimensional generalization of the W function; its importance will be exploited in the next section. On the other hand, the relation between the reference potential U_l and the target potential V can also be written as

$$U_l = V + \frac{l(l+1)}{r^2M} + \left(\frac{7}{16M^3}\left(\frac{dM}{dr}\right)^2 - \frac{1}{4M^2}\frac{d^2M}{dr^2} - \frac{1}{rM^2}\frac{dM}{dr}\right), \tag{16}$$

where the last term inside the parenthesis makes the difference with a one-dimensional problem [40–42].

To get exact solutions of the radial PDMSE it is necessary to look for point canonical transformations $u = u(r)$ that reduce the sum of the potential term V_l and of the point canonical transformation associated term V_{PCT}, to a known solvable potential $U_l(u)$. In this regard, the next section is devoted to analyze this condition for several point canonical transformations, some of which are cases analyzed in the current literature. However, before ending the present section, we consider some remarks about boundary conditions on the radial wave functions that follow from its previous expressions. First, in order to guarantee that function $R(r)$ be finite in the origin, the function $\chi = \varphi M^{1/4}$, that arose in Eq. 15 should satisfy the limiting condition $[\varphi M^{1/4}]_{r\to 0} = 0$; in the $M(0) \neq 0$ case it implies the boundary condition $\varphi(0) = 0$. The case when mass tends to zero at the origin means an increase of the radial interaction force when approaching the central point. The divergence of the real potential of the system at this point should lead to the boundary condition for the radial part of the wave function in the form $R(0)=0$.

On the other hand, the cancellation of the mass function at points different from the origin requires to consider the behavior of the potential term V_{PCT}. Let us propose that the mass function tends to zero according to $M \approx \varepsilon^k$, with $\varepsilon = |r - r_0|$ and $k>0$. Then, according to the equation that defines W, this function behaves like $W \approx -k\varepsilon^{-\frac{k+2}{2}}$, and the term of potential $V_{PCT} \approx k(3k/2+1)\varepsilon^{-(k+2)}$, which means that V_{PCT} has a divergence to a certain radius $r\neq 0$. In a similar form, if the mass tends to be null in infinite, according to the rule $M \approx r^{-k}$, with $k>0$, then $W \approx (k/4+1)r^{k/2-1}$, and consequently $V_{PCT} \approx 3k/4(k/4+1)r^{k-2}$. That is, when the mass tends to zero more quickly than r^{-2} then the potential V_{PCT} diverges. By a similar analysis, if the mass tends toward infinity as $M \approx r^k$ then the potential becomes of the form $V_{PCT} \approx r^{-k-2}$. These considerations allow to propose and analyze the character of the potentials that are solved in the applications.

Finally, we mention that in a similar way to the one-dimensional case, the wave function normalizations can be performed as much in the variable r as in variable u according to

$$\int r^2|R(r)|^2dr = \int |\varphi|^2M^{1/2}dr = \int |\varphi|^2du, \tag{17}$$

supposed that the angular functions $Y_{lm}(\theta, \varphi)$ are normalized as is usual.

Solvable three-dimensional PDMSE with spherical symmetry

Equations 6 and 14 show the interrelation that exists between the function of mass M, the function W, and the change of variable $u = u(r)$ of the PCT; anyone of these functions determines the term V_{PCT} in the relation of potentials expressed in Eq. 13. In this section we explore the possibilities of the function of mass or of the PCT to contribute to the potential through the part V_{PCT}, which together with the term $V_l(r)$ should generate a solvable potential U_l. To improve analysis we next list interrelation equations between the three mentioned functions in a detailed form. The task could be performed by using the variable r or the variable u:

From $u = u(r)$ and by defining

$$\frac{1}{f} = \frac{du}{dr}, \tag{18}$$

we have

$$M(r) = \left(\frac{du}{dr}\right)^2 = \frac{1}{f^2(r)} \tag{19}$$

and

$$W(r) = \frac{1}{\sqrt{M}}\frac{d}{dr}\ln\frac{r}{M^{1/4}} = \frac{f}{r} + \frac{1}{2}\frac{df}{dr}. \tag{20}$$

The PCT potential term V_{PCT} is given by

$$V_{PCT} = W^2 + f\frac{d}{dr}W(r). \tag{21}$$

Besides, the PCT is determined from $W = W(r)$ by

$$f(r) = \frac{2}{r^2}\left[\int r^2W(r)dr + C_1\right] \tag{22}$$

and

$$u(r) = \int \frac{dr}{f(r)} + C_2, \tag{23}$$

where C_1 and C_2 are, as usual, integration constants.

On the other hand, from $r = r(u)$ we first get

$$f(u) = \frac{dr}{du}, \quad M = \frac{1}{f^2(u)} \tag{24}$$

then

$$W(u) = \frac{d}{du} \ln \frac{r(u)}{M^{1/4}} = \frac{d}{du} \ln\left(r(u)\sqrt{f(u)}\right) \tag{25}$$

and the PCT potential term is

$$V_{PCT} = W^2 + W'(u). \tag{26}$$

Finally, from $W = W(u)$ the PCT comes from

$$r^3(u) = C_3 \int e^{\int 2W(u)\, du} du + C_4, \tag{27}$$

where C_3, C_4 are, as before, integration constants.

Let us now analyze closely the main relation between potentials U_l and V expressed in Eq. 13. After using expression (20) one finds that the centrifugal term $l(l+1)/(r^2 M)$ is similar to those included in the calculation of the V_{PCT} term, that is

$$\begin{aligned} U_l = V + l(l+1)\left(\frac{f}{r}\right)^2 \\ +2\frac{f}{r}\frac{df}{dr} + \frac{1}{4}\left(\frac{df}{dr}\right)^2 + \frac{1}{2} f \frac{d^2 f}{dr^2}, \end{aligned} \tag{28}$$

which shows the quadratic dependence of the centrifugal term and the V_{PCT} term on the function f and/or its derivatives. We consider this equation our second main result in this paper. Next we exploit some interesting applications for the proposed scheme.

Case I). $f(r) = \beta r$.

This is the simplest case in Eq. 28, it implies the mass function $M(r) = \beta^{-2} r^{-2}$ and the PCT given by $\beta u = 1\text{n}\, \beta r$ or $\beta r = \exp(\beta u)$. We also have $W = 3\beta/2$, $V_{PCT} = 9\beta^2/4$, consequently the relation of potentials becomes

$$U_l = V + l(l+1)\beta^2 + \frac{9}{4}\beta^2, \tag{29}$$

showing that the only difference between the potentials is a constant term which increases as the angular momentum does. The PCT is known to be the one allowing a relation between Coulomb potential and the Morse potential, meaning that it maps the radial domain into the full line range. We can propose any solvable unidimensional $V(u)$ in the sense of constant mass quantum mechanics to obtain a solvable $U_l(u)$. Let us list some of them

a) The Morse potential.

In the u variable, this potential is written as

$$V = D\left(e^{-2\alpha u} - 2\, e^{-\alpha u}\right) \tag{30}$$

that switches to the r variable by using $e^{-\alpha u} = \left(\frac{1}{\beta r}\right)^{\alpha/\beta}$ to get the solvable PDMSE potential

$$V = D\left(\left(\frac{1}{\beta^2 r^2}\right)^{\alpha/\beta} - 2\left(\frac{1}{\beta r}\right)^{\alpha/\beta}\right), \tag{31}$$

with eigenvalues

$$E_{nl} = -\frac{\alpha^2}{4}(\xi - 2n - 1)^2 + l(l+1)\beta^2 + \frac{9}{4}\beta^2, \tag{32}$$

n=0, 1, 2…, and normalized eigenfunctions

$$\begin{aligned} \varphi_n(w) = \sqrt{\frac{\alpha(\xi-2n-1)}{(\xi-n)}} \\ \times w^{\frac{\xi-2n-1}{2}} e^{-\frac{1}{2}w} L_n^{\xi-2n-1}(w), \end{aligned} \tag{33}$$

with $\xi = \frac{2\sqrt{D}}{\alpha}$ and $W = \xi e^{-\alpha u}$. The radial eigenfunctions turn to be

$$R_{nl}(r) = \varphi_n(w) \frac{1}{r\sqrt{\beta r}}, \tag{34}$$

with $w = \xi\left(\frac{1}{\beta r}\right)^{\alpha/\beta}$. A graph of this Morse-like potential and its eigenfunctions are presented on Fig. 1, where we observe that the discrete values of the energy have the usual relationship to the potential, except the last one, this energy eigenvalue is greater than the value of the

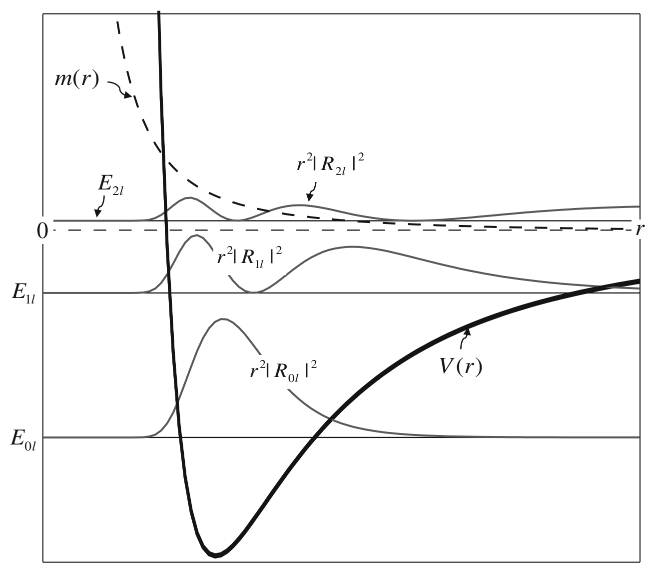

Fig. 1 This graph shows the radial Morse-like potential of Eq. 31 adjusted to be a potential with three eigenstates and the corresponding position-dependent mass distribution

potential at infinity. This situation seems to be characteristic of the PDMSE solutions as it has also been observed in some one-dimensional cases [49].

b) Harmonic Oscillator $V = \xi^2 u^2$.

In this case, βu = 1n βr leads to

$$V = \left(\frac{\xi}{\beta}\ln\beta r\right)^2 \tag{35}$$

with eigenvalues

$$E_{nl} = (2n+1)\xi + l(l+1)\beta^2 + \frac{9}{4}\beta^2 \tag{36}$$

and normalized eigenfunctions

$$R_{nl}(r) = \varphi_n(u)\frac{1}{r\sqrt{\beta r}} \tag{37}$$

where

$$\varphi_n(u) = \sqrt{\frac{\sqrt{\xi}}{2^n n!\sqrt{\pi}}}e^{-\xi u^2/2}H_n\left(\sqrt{\xi}u\right). \tag{38}$$

In Fig. 2, this potential and its corresponding radial eigenfunctions and mass distribution are shown. With regard to this figure and the earlier we can mention another situation, which has also been observed in one-dimensional PDMSE solutions, and that is, when mass values decrease, wave functions are distributed more widely, in such a way that the particle occupies a larger space but with low values of probability.

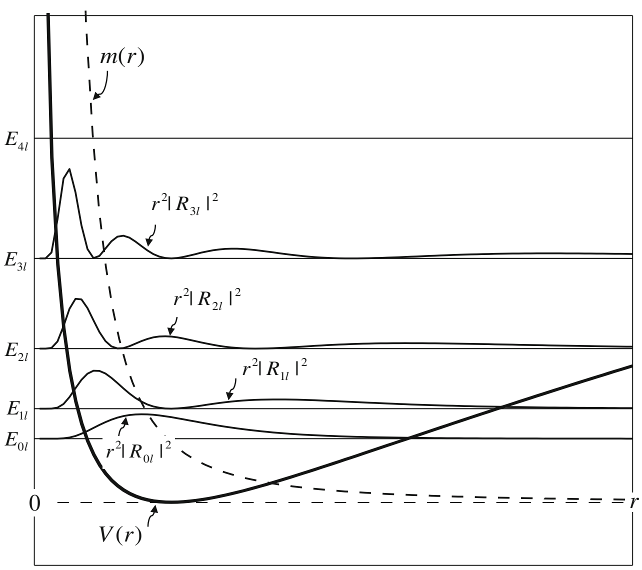

Fig. 2 This graph shows the radial Harmonic-like potential of Eq. 35 and the corresponding mass function

Case II). $f = r^{-\gamma/2}/\beta(1+\gamma/2)$

For this selection, every term in Eq. 28 is expressed in the same power of the variable r, which is a remarkable simplification. The mass has the form $M(r) = \alpha r^{\gamma}$, where $\alpha = \beta^2(1+\gamma/2)^2$ with $\beta > 0$. The change of variable is $\beta u = r^{1+\frac{\gamma}{2}}$, which maps the half r-line into the half u-line for a wide interval of values of γ. The W function is $W = (1-\gamma/4)/u(1+\gamma/2)$ and relation between the potentials is

$$U_l = V + \frac{l(l+1) - (1-\gamma/4)\frac{3}{4}\gamma}{(1+\gamma/2)^2}\frac{1}{u^2}, \tag{39}$$

where the second term, if interpreted as an angular momentum term of the form $\lambda(\lambda+1)/u^2$, gives the value of the new parameter λ in terms of the previous ones into the form

$$\lambda = -\frac{1}{2} + \frac{\sqrt{4l(l+1) + (\gamma-1)^2}}{\gamma+2}. \tag{40}$$

To have explicit solutions we propose any solvable potential in the half u - line.

a) Three-dimensional harmonic oscillator $V = \omega^2 u^2/4$.

In terms of r this potential is written

$$V = \left(\frac{\omega}{2\beta}\right)^2 r^{\gamma+2}, \tag{41}$$

with eigenvalues $E_{nl} = \left(2n + \lambda + \frac{3}{2}\right)\omega$, that can be carried to

$$E_{nl} = \left(2n + 1 + \frac{\sqrt{4l(l+1) + (\gamma-1)^2}}{\gamma+2}\right)\omega. \tag{42}$$

in accordance with ref. [51]. The eigenfunctions are

$$\varphi_{nl}(u) = N_{nl}\, y^{(\lambda+1)/2}e^{-y/2}L_n^{\lambda+1/2}(y), \tag{43}$$

with $y = \omega u^2/2$, such that radial functions are

$$R_{nl}(r) = N_{nl}\ \varphi_{nl}(u(r))\frac{\sqrt{\beta(1+\gamma/2)}r^{\gamma/4}}{r}, \tag{44}$$

with standard normalizations constants N_{nl}.

b) Coulomb potential $V = -Z/u$.

In terms of the variable r the Coulomb potential is given by

$$V = -Z\beta r^{-1-\gamma/2}, \tag{45}$$

whose eigenvalues are

$$E_{nl} = -\frac{Z^2}{4(n+\lambda+1)^2} \tag{46}$$

or in terms of angular momenta as

$$E_{nl} = -\frac{Z^2}{4\left(n+\frac{1}{2}+\frac{\sqrt{4l(l+1)+(\gamma-1)^2}}{\gamma+2}\right)^2}. \tag{47}$$

The eigenfunctions will be obtained from

$$\varphi_{nl}(u) = N_n\, y^{\lambda+1} e^{-y/2} L_n^{2\lambda+1}(y) \tag{48}$$

by taking $y = Zu/(n + \lambda + 1)$. With these functions we get the radial ones by using Eq. 15.

Case III). $f(r) = \exp(\beta r)$.

Straightforwardly, the mass distribution for $\beta>0$ is

$$M(r) = e^{-2\beta r} \tag{49}$$

and the change of variable maps the half line into the interval $0<\beta u<1$

$$\beta u = 1 - e^{-\beta r}. \tag{50}$$

Following previous formulae we get

$$W = e^{\beta r}\left(\frac{1}{r}+\frac{\beta}{2}\right) \tag{51}$$

and the relation of potentials is

$$U_l = V + e^{\beta r}\left(\frac{l(l+1)}{r^2}+\frac{2\beta}{r}+\frac{3\beta^2}{4}\right), \tag{52}$$

which allows to solve for an l - dependent potential that eliminates the second term and uses a solvable potential in the specified interval V_s, of the form

$$V = V_s - e^{\beta r}\left(\frac{l(l+1)}{r^2}+\frac{2\beta}{r}+\frac{3\beta^2}{4}\right). \tag{53}$$

If for example we use the infinite well potential

$$V_s(u) = \begin{cases} 0, & \beta u < 1 \\ \infty, & \beta u > 1 \end{cases} \tag{54}$$

we get the solution reported in ref. [54].

Case IV). $f(r) = \exp(-\beta r)$.

If we want to use solvable QMSE potentials defined in an infinite length interval for an exponential type mass, it should be done in the context of an increasing exponential mass $M(r) = e^{2\beta r}$. In such a case the change of variable from r - ray onto half u - line is given by $\beta u = e^{\beta r} - 1$, the W function is $W = e^{-\beta r}\left(\frac{1}{r}-\frac{\beta}{2}\right)$ and the relation of potentials becomes

$$U_l = V + e^{-\beta r}\left(\frac{l(l+1)}{r^2}-\frac{2\beta}{r}+\frac{3\beta^2}{4}\right). \tag{55}$$

At this point, as previously mentioned, the above equation can be used to solve l - dependent potentials of the form

$$V = V_s - e^{-\beta r}\left(\frac{l(l+1)}{r^2}-\frac{2\beta}{r}+\frac{3\beta^2}{4}\right) \tag{56}$$

with $U_l = V_s$ being for example the Coulomb potential $V_s = Z/u = -Z\beta\,(e^{\beta r}-1)^{-1}$ or the three-dimensional harmonic potential $V_s = \alpha^2 u^2/2 = \alpha^2\left(e^{\beta r}-1\right)^2/2\beta^2$. Let us consider the first case but we do not consider an l - dependent potential. In its place we take

$$V = \frac{-Z\beta}{e^{\beta r}-1} + e^{-\beta r}\left(\frac{2\beta}{r}-\frac{3\beta^2}{4}\right), \tag{57}$$

which substituted in Eq. 55 gives

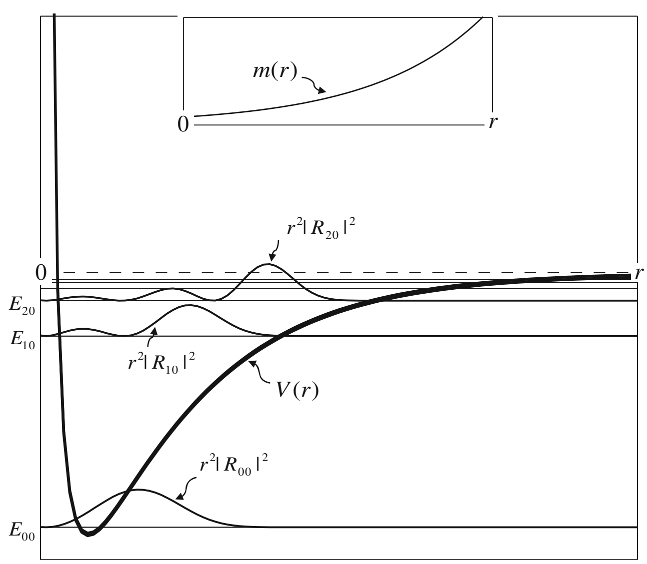

Fig. 3 This graph shows potential of Eq. 57, the corresponding mass function and the first eigenvalues with its radial probability densities

$$U_l = \frac{-Z\beta}{e^{\beta r}-1} + e^{-\beta r}\left(\frac{l(l+1)}{r^2}\right). \quad (58)$$

We can look for the *s* solutions of this case which match the Coulomb potential $U_0 = -Z\beta\left(e^{\beta r}-1\right)^{-1} = -Z/u$. The eigenvalues are

$$E_{n,l=0} = -\frac{Z^2}{4(n+1)^2} \quad (59)$$

and the eigenfunctions are constructed from the solutions given in Eq. 48 with λ=0. Figure 3 shows that the solved potential is a Morse-type potential featuring an infinite number of eigenstates, which is a notable property since it is usually known that a Morse potential has only a finite number of them.

Conclusions

The treatment for the resolution of the three-dimensional PDMSE problems with central potential has displayed some difficulties that can be appraised in set out literature. In this work we have elaborated a procedure to improve the existing ones and allow to generate new solvable cases. Although some solutions have been obtained and reported nowadays, some new solutions were discussed. At the end of each case we have included comments that could serve to evidence the physical properties of variable mass systems. The studied potentials are those that have a kind of relation with the harmonic or the Morse oscillator potentials. In each of the four considered cases, the relation between potentials of the reference and target problems are general enough to admit new solutions. To continue this work, one should take into account that new cases of the relationship between the above mentioned potentials are possible and they have the possibility of generating more solutions. Moreover, it will be possible to modify the scheme to deal with the Bohr Hamiltonian used in nuclear theory. In fact, the energy spectra obtained have properties that deserve comparison with those spectra generated in other aggregation problems, such as nuclear problems.

Acknowledgments G. O. and J. M. are thankful for the support received from CONACYT-MEXICO through the SNI program.

References

1. Bastard G (1988) Wave mechanics applied to semiconductor heterostructures. Editions de Physique, Les Ulis
2. Einevoll GT, Hemmer PC, Thomsen J (1990) Phys Rev B 42:3485, and references there in
3. Von Roos O, Mavromatis H (1985) Phys Rev B 31:2294
4. Weisbuch C, Vinter B (1993) Quantum semiconductor heterostructures. Academic, New York
5. Landsberg GT (1969) Solid state theory: methods and applications. Wiley, London
6. Renan R, Pacheco MH, Almeida CAS (2000) J Phys A 33(50): L509
7. Plastino AR, Casas M, Plastino A (2001) Phys Lett A 281:297
8. Ring P, Schuck P (1980) The nuclear many body poblem. Springer, Heidelberg, p 211
9. Arias de Saavedra F, Boronat J, Polls A, Fabrocini A (1994) Phys Rev B 50:4248
10. Puente A, Serra L, Casas M (1994) Z Phys D 31:283
11. Barranco M, Pi M, Gatica SM, Hernandez ES, Navarro J (1997) Phys Rev B 56:8997
12. Foulkes WMC, Schluter M (1990) Phys Rev B 42:11505
13. Bonatsos D, Georgoudis PE, Lenis D, Minkov N, Quesne C (2011) Phys Rev C 83:044321
14. Boztosun I, Bonatsos D, Inci I (2008) Phys Rev C 77:044302
15. Geller MR, Kohn E (1993) Phys Rev Lett 70:3103
16. Serra L, Lipparini E (1997) Europhys Lett 40:667
17. Barranco M, Hernández ES, Navarro J (1996) Phys Rev B 54:7394
18. Harrison P (2000) Quantum wells, wires and dots. Wiley, New York
19. Li YM, Lu HM, Voskoboynikov O, Lee CP, Sze SM (2003) Surf Sci 532:811
20. von Roos O (1983) Phys Rev B 27:7547
21. Marc J, Leblond L (1995) Phys Rev A 52:1845
22. de Souza Dutra A, Almeida CAS (2000) Phys Lett A 275:25
23. Bagchi B, Gorain P, Quesne C, Roychoudhury R (2004) Mod Phys Lett A 19:2765
24. Plastino AR, Puente A, Casas M, Garcias F, Plastino A (2000) Rev Mex Fis 46:78
25. Yu J, Dong SH (2004) Phys Lett A 325:194
26. Solimmanejad M, Moayedi SK, Tavakoli M (2006) Int J Quant Chem 106:1027
27. Tezcan C, Sever R (2007) J Math Chem 42:387
28. Tezcan C, Sever R (2008) Int J Theor Phys 47:1471
29. Jiang L, Yi LZ, Jia CS (2005) Phys Lett A 345:279
30. Bagchi B, Gorain P, Quesne C, Roychoudhury R (2005) Europhys Lett 72:155
31. Ju GX, Cai CY, Xiang Y, Ren ZZ (2007) Commun Theor Phys 47:1001
32. Roy B, Roy P (2002) J Phys A Math Gen 35:3961
33. Koc R, Koca M (2003) J Phys A Math Gen 36:8105
34. Roy B (2005) Europhys Lett 72:1
35. Dong SH, Peña JJ, Pacheco-García C, García-Ravelo J (2007) Mod Phys Lett A 22:1039
36. Plastino AR, Rigo A, Casas M, Garcias F, Plastino A (1999) Phys Rev A 60:4318
37. Milanovic V, Ikovic Z (1999) J Phys A Math Gen 32:7001
38. Gönül B, Gönül B, Tutcu D, Özer O (2002) Mod Phys Lett A 17:2057
39. Quesne C (2006) Ann Phys 321:1221
40. Aktas M, Sever R (2007) J Math Chem 43:92
41. Bagchi B, Gorain P, Quesne C, Roychoudhury R (2004) Czech J Phys 54:1019
42. Gönül B, Özer O, Gönül B, Üzgün F (2002) Mod Phys Lett A 17:2453
43. Bagchi B, Bannerjee A, Quesne C, Tkachuk VM (2005) J Phys A Math Gen 38:2929
44. Quesne C, Bagchi B, Banerjee A, Tkachuk VM (2006) Bulg J Phys 33:308
45. Ganguly A, Nieto LM (2007) J Phys A Math Gen 40:7265
46. Peña JJ, Morales J, Zamora-Gallardo E, García-Ravelo J (2004) Int J Quant Chem 100:957

47. Peña JJ, Ovando G, Morales J, García-Ravelo J, Pacheco-García C (2008) Int J Quant Chem 108:2906
48. Morales J, Ovando G, Peña JJ (2010) Symmetries in nature. Symposium in memoriam Marcos Moshinsky. Am Inst Phys AIP-EUA 1323:233
49. Ovando G, Peña JJ, Morales J (2011) Progress in theoretical chemistry and physics, vol 22. Springer, Heidelberg, pp 33
50. Peña JJ, Ovando G, Morales J, García-Ravelo J, Pacheco-García C (2007) Int J Quant Chem 107:3039
51. Alhaidari AD (2002) Phys Rev A 66:042116
52. Yu J, Dong SH, Sun GH (2004) Phys Lett A 322:290
53. Gang C (2004) Phys Lett A 329:22
54. Cai CY, Ren ZZ, Ju GX (2005) Commun Theor Phys (Beijing, China) 43:1019